

6.4110/16.420  
Representation, Inference and Reasoning in AI

Quiz 1

**Solutions**

February 25, 2026

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

You are permitted to use a single sheet of paper with notes on (both sides). You may not use a calculator. **Box all answers** for free response questions.

Name: \_\_\_\_\_

MIT email: \_\_\_\_\_

Question	Points	Score
1	27	
2	39	
3	34	
Total:	100	

## McDolphin



1. Assume you have a set of employees  $e \in E$ , restaurant locations  $l \in L$  within a town, and discrete time slots  $h \in H$  over a week. Each employee  $e$  has a set of locations they are able to work at  $L_e \subseteq L$  and a set of time slots during which they are available  $H_e \subseteq H$ . Each location  $l$  requires staffing during a specified subset of time slots  $H_l \subseteq H$ . A location is staffed at a given time if **exactly one** employee is working there, though an employee could work more than one hour at more than one location total. Your goal is to find a staffing schedule so that every location is staffed at each hour it is open.
- (a) For each of the three possible formulations below, indicate the domains of the variables so that an assignment between **each variable and a single element** of the domain would constitute a possible solution to the problem. No formulation should place additional constraints on valid assignments beyond those described above. If the answer is “None of these,” explain in the box below, select the valid domain, and leave the box blank.
- i. (3 points) One variable  $V_{l,h}$  for each location  $l \in L$  and hour  $h \in H_l$   
  $E$      $H$      $L$      $H \times L$      $H \times E$      $E \times L$     None of these (Explain below)

**Solution:**

One can assign an employee to each hour and location and solve the problem.

- ii. (3 points) One variable  $V_l$  for **each location**  $l \in L$ .  
  $E$      $H$      $L$      $H \times L$      $H \times E$      $E \times L$     **None of these (Explain below)**

**Solution:**

For a given location we could have multiple people working at different times over the course of the week. This wouldn't be possible in this assignment.

- iii. (3 points) One variable  $V_{l,e}$  for each employee  $e \in E$  and location  $l \in L_e$ .  
  $E$      $H$      $L$      $H \times L$      $H \times E$      $E \times L$     **None of these (Explain below)**

**Solution:** Any employee might work multiple hours over the course of the week which can't be represented with this variable specification or these options.

- (b) For the problems below, express the constraints using these variables:

$E, H, L, L_e, H_e, H_l, e, h, l, V_{l,h}, Pat (P), Plainville (N),$  and  $3pm\ Thursdays (3pmT)$ .

If you find it useful, you may introduce additional variables or domains derived from these, but you must clearly and explicitly define anything new you use. Every answer must be written in set notation; we will allow some flexibility in how you specify the sets, provided they are clearly and correctly defined.

- i. (4 points) In the formulation with one variable  $V_{l,h}$  for each location and hour, what are the unary constraints? Recall that if a constraint is already satisfied by variable specification, it should not be a constraint.

**Solution:**

We were flexible about the format of your answer so here are a few acceptable solutions:

1.  $c_{l,h}^1 = \langle (V_{l,h}), \{e \in E \mid h \in H_e\} \rangle$  and  $c_{h,l}^2 = \langle (V_{l,h}), \{e \in E \mid l \in L_e\} \rangle$ .
2.  $c_{l,h}^3 = \langle (V_{l,h}), \{e \in E \mid h \in H_e, l \in L_e\} \rangle$
3.  $c_{l,h}^4 = \langle (V_{l,h}), R_{l,h} \rangle$  where  $R_{l,h} \subseteq E \iff l \in L_e \wedge h \in H_e$
4.  $\{e \in E \mid h \in H_e, l \in L_e\}$

- ii. (4 points) In the formulation with one variable  $V_{l,h}$  for each location and hour, we need to specify some binary constraints. For two variables  $V_{l_1,h_1}$  and  $V_{l_2,h_2}$ , provide an expression for the binary constraint between them.

**Solution:**

We need to define a constraint that prevents one person from being in multiple locations at once. We were flexible about the format of your answer, so here are a few acceptable solutions:

$Different(l_1, h_1, e_1, l_2, h_2, e_2) := (h_1 \neq h_2) \vee (l_1 = l_2) \vee (e_1 \neq e_2)$

This means,  $e_1$  and  $e_2$  are valid if they are a different employee. Assuming they aren't a different employee, then if they are at the same location, that is okay, since we have a unique variable for every hour at every location. Finally, if two assignments are the same employee and are at different locations, then that could be fine so long as it isn't at the same time.

1.  $c_{l_1,h_1,l_2,h_2}^5 = \langle (V_{l_1,h_1}, V_{l_2,h_2}), \{(e_1, e_2) \in E \times E \mid Different(l_1, h_1, e_1, l_2, h_2, e_2)\} \rangle$ .
2.  $c_{l_1,h_1,l_2,h_2}^6 = \langle (V_{l_1,h_1}, V_{l_2,h_2}), R^4 \rangle$  so that  $R^4 \subseteq E \times E \iff Different(l_1, h_1, e_1, l_2, h_2, e_2)$ .

- iii. (4 points) What constraint would you add to represent that an employee Pat is unable to staff Plainville at 3PM on Thursdays?

**Solution:**

1.  $c_{l,h}^7 = \langle (V_{N,3pmT}), \{e \in E - \{P\}\} \rangle.$

2.  $c_{l,h}^8 = \langle (V_{N,3pmT}), \{e \in E \mid e \neq P\} \rangle.$

3.  $V_{N,3pmT} \neq Pat$

(c) (6 points) Consider a problem instance with 2 locations (**P**lainville, **C**ambridge), 2 hours (**8am** and **6pm** on Monday), and 3 employees (**A**lex, **B**laine, and **D**elilah), where both locations are open during both hours. Assume the problem formulation where we have variables  $V_{l,h}$  for each location and hour. All three employees are students at MIT.

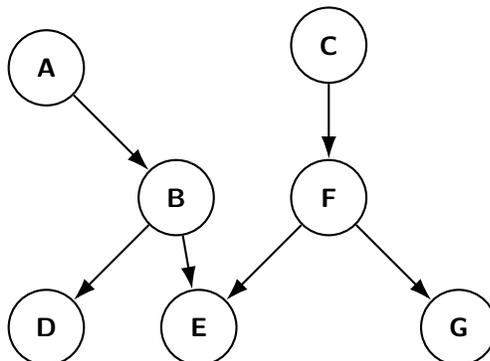
1. **A** takes classes in the afternoon, so could only work at **P** in the morning, but could make it to the location in **C** at any time of day.
2. Since **B** never wakes up before 1pm she is unable to work in the morning.
3. **A** babysits **B**'s pet fish so can't work during any time that **B** is working.
4. **D** is very flexible on time and location but is only willing to work a single hour a week because she is hosed with problem sets.

Using the formulation with one variable per location, hour pair, assuming variable ordering  $V_{C,6pm}$ ,  $V_{C,8am}$ ,  $V_{P,6pm}$ ,  $V_{P,8am}$ , if we assign **B to work at C at 6pm**, which possible assignments does forward checking eliminate? Respond by listing the contents of the domain for each variable before and after this assignment.

Variable	Domain (before assigning $B$ to $V_{C,6PM}$ )	Domain (after assigning $B$ to $V_{C,6PM}$ and FC that assignment)
$V_{C,6pm}$	$\{A, B, D\}$	$\{B\}$
$V_{C,8am}$	$\{A, D\}$	$\{A, D\}$
$V_{P,6pm}$	$\{B, D\}$	$\{D\}$
$V_{P,8am}$	$\{A, D\}$	$\{A, D\}$

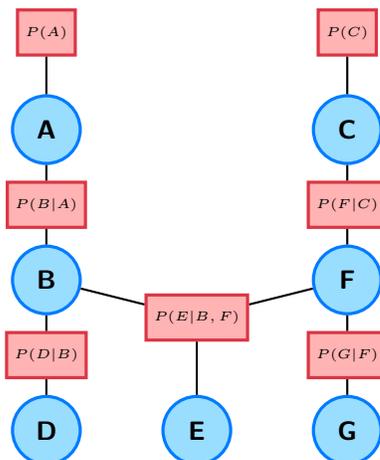
## Fun with Factors

2. Consider the following Bayesian network:

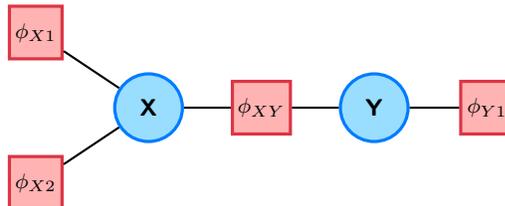


- (a) (2 points) Is it a polytree?  
 **Yes**    No
- (b) (3 points) Is it possible to express any joint distribution on these 7 variables using this network structure?  
 Yes    **No**
- (c) (3 points) Imagine that we just found out the value of D and A. Does this tell us anything about C?  
 Yes    **No**
- (d) (3 points) Say we already know the value of F. If we additionally learn the value of E, does this tell us anything more about what C might be?  
 Yes    **No**
- (e) (3 points) Say we learned the value of A. Given this information, is B independent of F?  
 **Yes**    No
- (f) (3 points) Select all variables that, if observed, would make A and C dependent.  
 A    B    C    D    E    F    G    There is no such variable
- (g) (6 points) Draw the Bayesian network above as an undirected factor graph.

**Solution:**



- (h) (6 points) Consider a new factor graph with two binary variables,  $X \in \{0, 1\}$  and  $Y \in \{0, 1\}$ . The graph contains one pairwise factor  $\phi_{XY}$  connecting them, two unary factors  $\phi_{X1}$  and  $\phi_{X2}$  connected to  $X$ , and one unary factor  $\phi_{Y1}$  connected to  $Y$ .



The factor tables are defined as follows:

$X$	$\phi_{X1}(X)$
0	2
1	1

$X$	$\phi_{X2}(X)$
0	1
1	3

$Y$	$\phi_{Y1}(Y)$
0	3
1	1

$X$	$Y$	$\phi_{XY}(X, Y)$
0	0	1
0	1	2
1	0	2
1	1	1

Calculate the exact marginal probabilities  $P(X)$  and  $P(Y)$ .

**Solution:** First, we compute the unnormalized joint distribution  $\tilde{P}(X, Y) = \phi_{X1}(X) \cdot \phi_{X2}(X) \cdot \phi_{Y1}(Y) \cdot \phi_{XY}(X, Y)$ .

- $\tilde{P}(0, 0) = 2 \times 1 \times 3 \times 1 = 6$
- $\tilde{P}(0, 1) = 2 \times 1 \times 1 \times 2 = 4$
- $\tilde{P}(1, 0) = 1 \times 3 \times 3 \times 2 = 18$
- $\tilde{P}(1, 1) = 1 \times 3 \times 1 \times 1 = 3$

Next, compute the partition function (normalizing constant)  $Z$ :

$$Z = 6 + 4 + 18 + 3 = 31$$

Now, marginalize and normalize to find  $P(X)$  and  $P(Y)$ :

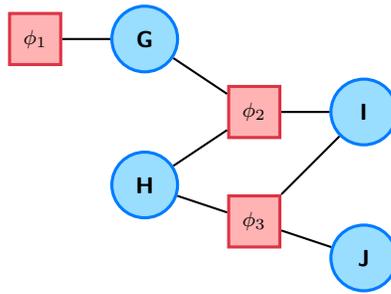
- $P(X = 0) = \frac{6+4}{31} = \frac{10}{31}$
- $P(X = 1) = \frac{18+3}{31} = \frac{21}{31}$
- $P(Y = 0) = \frac{6+18}{31} = \frac{24}{31}$
- $P(Y = 1) = \frac{4+3}{31} = \frac{7}{31}$

**Final Marginals:**

$$P(X) = [10/31, 21/31]$$

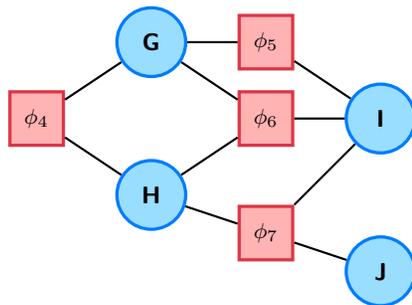
$$P(Y) = [24/31, 7/31]$$

- (i) (10 points) Quinn is analyzing the factor graph below, but realizes that exact Belief-Propagation can't work since there is a loop.



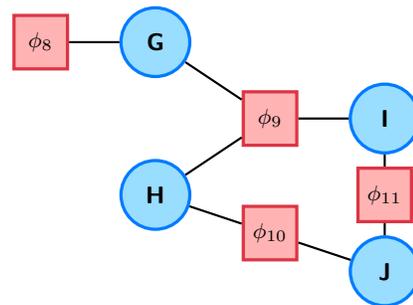
For each of the following modified factor graphs, indicate whether it could be made **equivalent** to the original graph with a careful selection of new factors, and whether exact **Belief-Propagation** could be successfully run on it.

Graph A



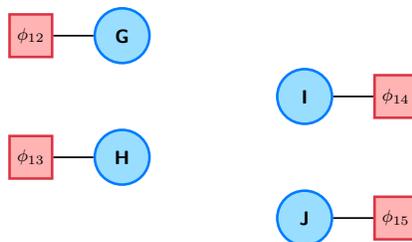
- Distribution:**  
 Could be Equivalent     Not Equivalent
- Exact BP Works?**  
 Yes     No

Graph B



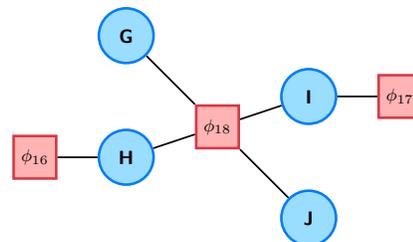
- Distribution:**  
 Could be Equivalent     Not Equivalent
- Exact BP Works?**  
 Yes     No

Graph C



- Distribution:**  
 Could be Equivalent     Not Equivalent
- Exact BP Works?**  
 Yes     No

Graph D

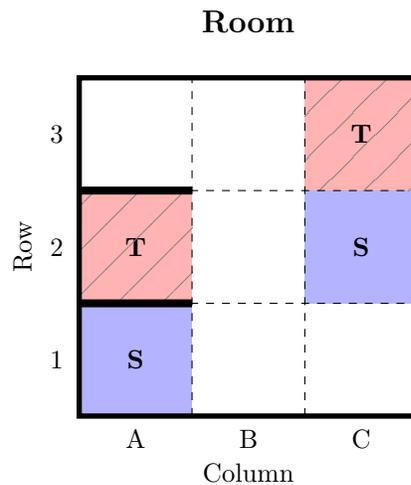


- Distribution:**  
 Could be Equivalent     Not Equivalent
- Exact BP Works?**  
 Yes     No

## Dungeon Crawler

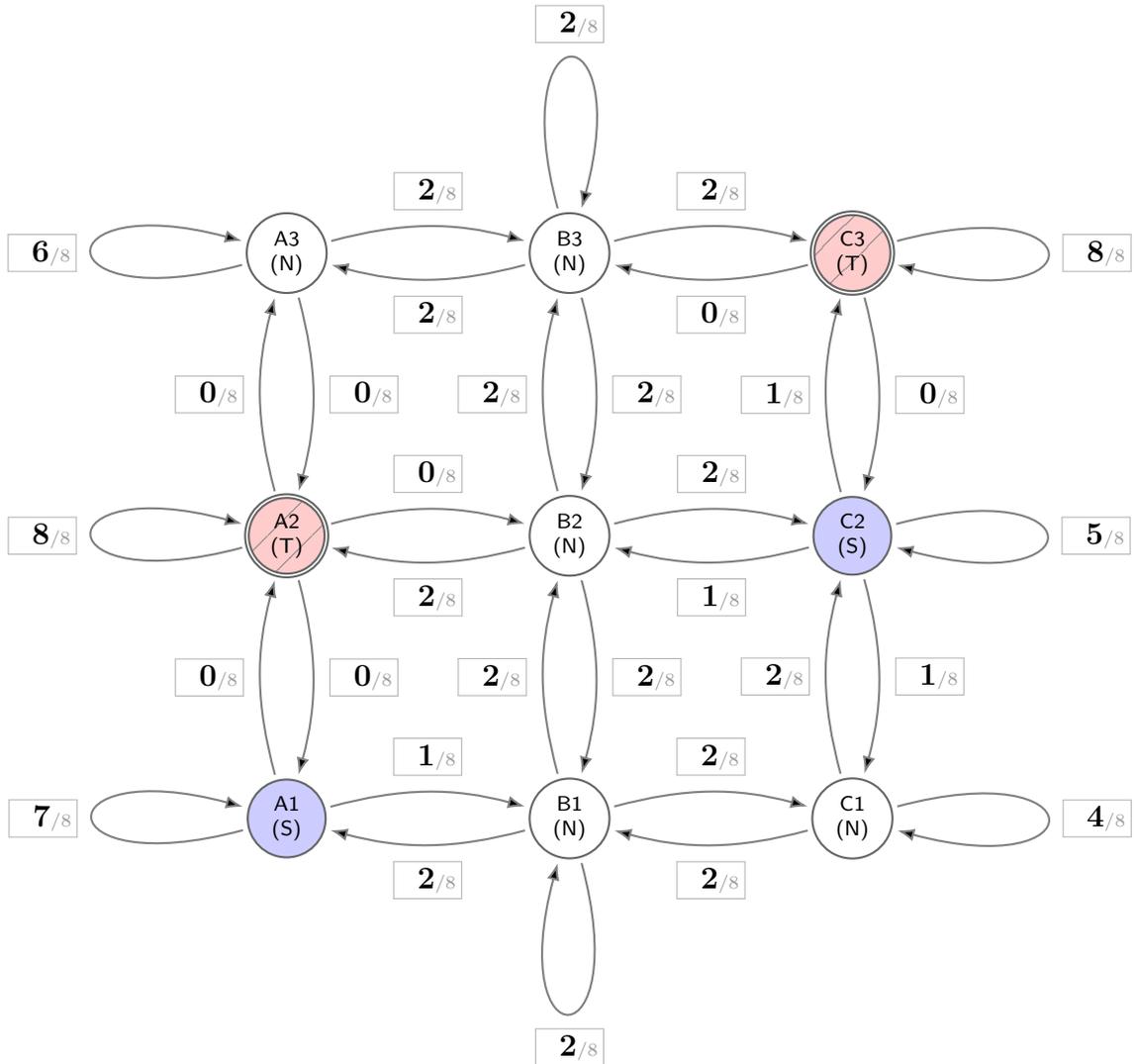
3. You are playing a text-based RPG called *Dungeon Crawler*. The game is notoriously unfair and relies entirely on random moves.

- The map is a  $3 \times 3$  grid of tiles with rows  $\in \{1, 2, 3\}$  and columns  $\in \{A, B, C\}$ .
- You start at  $t = 0$  on a random tile  $X_0$ . Each tile is equally likely.
- At every time step, the player moves in a random direction that depends on the type of tile they are in. There are three types of tiles:
  - Normal (**N**): Equal probability of moving *up*, *down*, *left* or *right*. All unlabeled tiles are normal.
  - Stuck (**S**): There is a 50% chance of not moving. Otherwise, the player moves randomly like **N**.
  - Trap (**T**): The player never moves from the trap state once entered.
- **Walls:** If you attempt to move into a wall (a thick black line), you stay in your current square for that time step instead of moving.
- **Observations:** You can see only the type of tile you are standing on (**N**, **S**, or **T**). Your observation of your tile type at  $X_t$  is  $O_t$ .



Assume now that we are in the room above. Note that there are walls on three sides of the **T** in tile **2A**, meaning a player can only enter that **T** tile from location **2B**, but the **T** in tile **3C** can be entered from **B3** or **C2**. Note: None of the problems in this section require matrix multiplication to solve.

- (a) (10 points) Finish the HMM transition diagram that represents the room by drawing all the other transitions between nodes and their respective probabilities. **We have already filled in the denominator of 8 for all probabilities, so so only write numerator.** For example, the probability of transitioning from A3 to A3 would be  $\frac{6}{8}$ , so we would write 6 on this diagram. We have already done all the transitions to and from A3 for you. Fill in the rest of the diagram.



- (b) (2 points) At  $t = \infty$  what is the probability you have fallen into a trap?

**Solution:** 1.0

(c) You have survived for two turns in the room and experience the following observation sequence:

$$O_0 = \mathbf{S}, \quad O_1 = \mathbf{S}, \quad O_2 = \mathbf{N}$$

i. (2 points) What is your belief state after  $O_0$ ? That is,  $P(X_0 | O_0 = \mathbf{S})$ ?

**Solution:**

$$P(X_0 = A1 | O_0 = \mathbf{S}) = \frac{1}{2}, \quad P(X_0 = C2 | O_0 = \mathbf{S}) = \frac{1}{2},$$

and  $P(X_0 = c | O_0 = \mathbf{S}) = 0$  for all other tiles  $c$ .

ii. (2 points) And then, after you take another step? That is,  $P(X_1 | O_0 = \mathbf{S}, O_1 = \mathbf{S})$ ?

**Solution:**

Since the two S tiles are not adjacent, observing S at  $t = 1$  implies  $X_1 = X_0$ .

$$P(X_1 = A1 | X_0 = A1) = \frac{7}{8}, \quad P(X_1 = C2 | X_0 = C2) = \frac{5}{8}.$$

$$P(A1 | O_0 = \mathbf{S}, O_1 = \mathbf{S}) = \frac{7}{12}$$

$$P(C2 | O_0 = \mathbf{S}, O_1 = \mathbf{S}) = \frac{5}{12}$$

0 otherwise.

iii. (2 points) And then, after you take another step? That is,  $P(X_2 | O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N})$ ?

**Solution:** From the previous part,

$$P(X_1 = A1 | O_0 = \mathbf{S}, O_1 = \mathbf{S}) = \frac{7}{12}, \quad P(X_1 = C2 | O_0 = \mathbf{S}, O_1 = \mathbf{S}) = \frac{5}{12}.$$

Now condition on  $O_2 = \mathbf{N}$ . The only N-states reachable in one step are:

$$A1 \rightarrow B1 \quad (\text{prob } 1/8),$$

$$C2 \rightarrow B2 \quad (\text{prob } 1/8), \quad C2 \rightarrow C1 \quad (\text{prob } 1/8).$$

So the unnormalized probabilities are

$$P(X_2 = B1, \text{obs}) \propto \frac{7}{12} \cdot \frac{1}{8} = \frac{7}{96},$$

$$P(X_2 = B2, \text{obs}) \propto \frac{5}{12} \cdot \frac{1}{8} = \frac{5}{96},$$

$$P(X_2 = C1, \text{obs}) \propto \frac{5}{12} \cdot \frac{1}{8} = \frac{5}{96}.$$

Normalizing by

$$\frac{7 + 5 + 5}{96} = \frac{17}{96},$$

we get

$$P(X_2 = x | O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N}) = \begin{cases} \frac{7}{17}, & x = B1, \\ \frac{5}{17}, & x = B2, \\ \frac{5}{17}, & x = C1, \\ 0, & \text{otherwise.} \end{cases}$$

iv. (3 points) What is  $P(X_1 | O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N})$ ?

**Solution:** We use smoothing:

$$P(X_1 \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N}) \propto P(O_2 = \mathbf{N} \mid X_1) P(X_1 \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}).$$

From the previous part,

$$P(X_1 = A1 \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}) = \frac{7}{12}, \quad P(X_1 = C2 \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}) = \frac{5}{12},$$

and all other states have probability 0.

Now compute the likelihood of observing **N** at time 2:

$$P(O_2 = \mathbf{N} \mid X_1 = A1) = \frac{1}{8}$$

(since  $A1 \rightarrow B1$  is the only move to an **N**-tile), and

$$P(O_2 = \mathbf{N} \mid X_1 = C2) = \frac{2}{8} = \frac{1}{4}$$

(since  $C2 \rightarrow B2$  or  $C2 \rightarrow C1$ , each with probability  $1/8$ ).

Thus the unnormalized values are

$$P(X_1 = A1 \mid \dots) \propto \frac{7}{12} \cdot \frac{1}{8} = \frac{7}{96}, \quad P(X_1 = C2 \mid \dots) \propto \frac{5}{12} \cdot \frac{2}{8} = \frac{10}{96}.$$

Normalizing by  $\frac{7+10}{96} = \frac{17}{96}$ , we get

$$P(X_1 = x \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N}) = \begin{cases} \frac{7}{17}, & x = A1, \\ \frac{10}{17}, & x = C2, \\ 0, & \text{otherwise.} \end{cases}$$

v. (3 points) What is  $P(X_0 \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N})$ ?

**Solution:** Since  $O_0 = \mathbf{S}$ , the only possible initial states are  $A1$  and  $C2$ , and

$$P(X_0 = A1 \mid O_0 = \mathbf{S}) = P(X_0 = C2 \mid O_0 = \mathbf{S}) = \frac{1}{2}.$$

Now compute the likelihood of the remaining observations  $O_1 = \mathbf{S}, O_2 = \mathbf{N}$  from each start state.

If  $X_0 = A1$ , then to observe  $\mathbf{S}$  at  $t = 1$  and  $\mathbf{N}$  at  $t = 2$ , the only possible path is

$$A1 \rightarrow A1 \rightarrow B1,$$

with probability

$$P(O_1 = \mathbf{S}, O_2 = \mathbf{N} \mid X_0 = A1) = \frac{7}{8} \cdot \frac{1}{8}.$$

If  $X_0 = C2$ , then to observe  $\mathbf{S}$  at  $t = 1$  and  $\mathbf{N}$  at  $t = 2$ , we must have

$$C2 \rightarrow C2 \rightarrow B2 \quad \text{or} \quad C2 \rightarrow C2 \rightarrow C1,$$

so

$$P(O_1 = \mathbf{S}, O_2 = \mathbf{N} \mid X_0 = C2) = \frac{5}{8} \cdot \frac{2}{8}.$$

Thus,

$$P(X_0 = A1 \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N}) \propto \frac{1}{2} \cdot \frac{7}{8} \cdot \frac{1}{8} = \frac{7}{128},$$

$$P(X_0 = C2 \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N}) \propto \frac{1}{2} \cdot \frac{5}{8} \cdot \frac{2}{8} = \frac{10}{128}.$$

Normalizing by

$$\frac{7 + 10}{128} = \frac{17}{128},$$

we get

$$P(X_0 = x \mid O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N}) = \begin{cases} \frac{7}{17}, & x = A1, \\ \frac{10}{17}, & x = C2, \\ 0, & \text{otherwise.} \end{cases}$$

(d) (10 points) What is the **most likely sequence** of states (path)  $X_0, X_1, X_2$  that explains these observations? If there are multiple equally likely paths for the observations  $O_0 = \mathbf{S}, O_1 = \mathbf{S}, O_2 = \mathbf{N}$ , list one of them.

**Solution:** The only state paths consistent with  $\mathbf{S}, \mathbf{S}, \mathbf{N}$  are:

$$A1 \rightarrow A1 \rightarrow B1, \quad C2 \rightarrow C2 \rightarrow C1, \quad C2 \rightarrow C2 \rightarrow B2.$$

Their (unnormalized) probabilities given all the observations are:

$$P(A1 \rightarrow A1 \rightarrow B1) = \frac{1}{2} \cdot \frac{7}{8} \cdot \frac{1}{8} = \frac{7}{128},$$

$$P(C2 \rightarrow C2 \rightarrow C1) = \frac{1}{2} \cdot \frac{5}{8} \cdot \frac{1}{8} = \frac{5}{128}, \quad P(C2 \rightarrow C2 \rightarrow B2) = \frac{1}{2} \cdot \frac{5}{8} \cdot \frac{1}{8} = \frac{5}{128}.$$

Therefore the most likely path is

$$\boxed{X_0 = A1, X_1 = A1, X_2 = B1.}$$