L04 – Inference in Undirected Trees

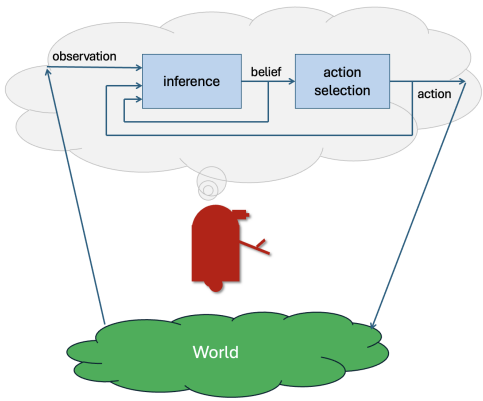
Barber 4.1, 4.4, 5.1,5.2

What you should know after this lecture

- Belief-propagation algorithm for inference in tree-structured factor graphs
- Max-product to find maximum-likelihood assignment

Probabilistic belief representation

- Belief is a probability distribution : $B \in \mathcal{P}(\mathcal{S})$
  (an element of the set of all distributions over $\mathcal{S}$)
- Important questions:
  - What is $p_B(\text{event})$?
  - What is the most likely state $\text{argmax}_s\, p_B(s)$?
  - How should we update B given an observation?



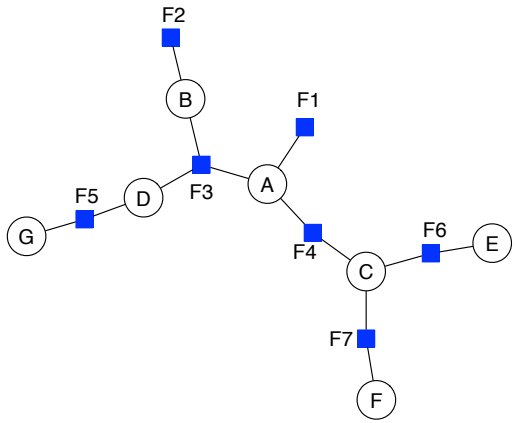
Factored representation of B

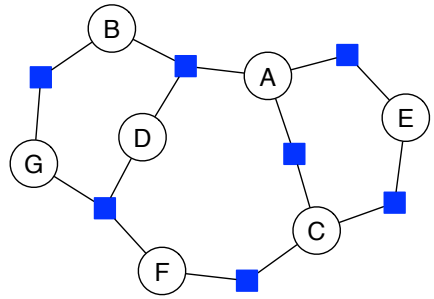
- Random variables $V_1, \ldots, V_n$
- Each V_i has discrete domain of possible values Ω_{V_i}
- Sample space is product $\mathcal{S} = \Omega_{V_1} \times \ldots \times \Omega_{V_n}$
- Sample $s \in \mathcal{S}$ is $(v_1, \ldots, v_n)$ where $v_i \in \Omega_{v_i}$
- B is the joint distribution on $V_1, \ldots, V_n$
- Can use a table α to represent B
- Use Boolean expressions over atoms $V = v$ to represent Q and E

Undirected models

- Directed models (Bayes nets) are good for many problems, particularly when there is a causal interpretation of the arrows. (Though causality is not necessary)
- Relationship between pixels in an image or adjacent plots of property is not independent but there's no sensible way to assign a direction.
- Can make graphical models with nodes and undirected arcs: Markov random fields
- We will skip that step and go straight to a formalism called factor graphs that can represent both directed and undirected models.
- A generalization of factor graphs for CSPs!!

Factor graphs





Undirected bipartite graph: factors only connect to variables

- Round nodes are random variables V
- Square nodes are factors ϕ: tables specifying, for each tuple of value of the connected variables, a non-negative number
- Represent a probability distribution (e.g. left graph above)

$$P((a,b,c,d,e,f)) = \frac{1}{Z}\phi_1(a)\phi_2(b)\phi_3(a,b,d)\phi_4(a,c)\phi_5(d,g)\phi_6(c,e)\phi_7(c,f)$$

where Z is a normalizer

$$Z = \sum_{a,b,c,d,e,f} \phi_1(a)\phi_2(b)\phi_3(a,b,d)\phi_4(a,c)\phi_5(d,g)\phi_6(c,e)\phi_7(c,f)$$

PGMs and CSPs

- In both PGMs and CSPs, the nodes represent variables, with finite domains.
- The factors are tables of values, one for each assignment of the variables they are connected to.
- In CSPs, the values have to be 0 and 1.
- In PGMs, the values can be any non-negative number.
- The factor graph of a CSP defines a **set** of assignments.
- The factor graph of a PGM defines a **distribution** over assignments.

Independence relations in factor graphs

- The Markov blanket of a node V consists of all nodes that are connected to any factor connected to V.
- The Markov blanket of A in our example is $\{B, D, C\}$
- A node V is not, in general, independent of any node in its MB
- A node V is conditionally independent of the rest of the graph, conditioned on $\text{mb}(V)$
- There are some sets of independence relations that are describable by a Bayes net but not describable by a factor graph (and vice versa)

Inference in factor graphs

Some inference problems:

- Joint distribution: In a factor graph, use table multiplication to compute a big table

$$\frac{1}{Z}\prod_{k}\phi_k$$

  where Z is the sum of all table entries
- Marginal distribution: $P(Y)$ where $Y \subset \mathcal{V}$
- Conditional probability: $P(Y \mid E = e)$, where $Y \subset \mathcal{V}$, $E \subset \mathcal{V}$, and $Y \cap E = \emptyset$; and e is the observed values of the variables in E. Note that it is not necessary that $Y \cup E = \mathcal{V}$.
- Most probable assignment (MAP):

$$\operatorname{argmax}_y P(Y = y \mid E = e) \quad .$$

  Note that the MAP of a set of variables is not necessarily the set of MAPs of the individual variables.

Computing all the individual marginals

- This method only applies if your factor graph does not have any cycles!
- Awesome algorithm with many names: belief propagation, sum-product, message passing
- Runs in time $O(N \cdot |T^*|)$ where N is the number of nodes and $|T^*|$ is number of entries in the largest table (exponential in the number of variables it is connected to).
- Can parallelize the computation.

Belief propagation idea

- Pick an arbitrary variable $V_i \in \mathcal{V}$ to be the root node
- Let $N(V)$ be the factors connected to V, $N(\phi)$ vars connected to ϕ

$$\begin{aligned} P(v_i) \propto \sum_{\bar{v}_{\mathcal{V} \setminus V_i}} P(\bar{v}) &= \sum_{\bar{v}_{\mathcal{V} \setminus V_i}} \prod_j \phi_j(\bar{v}) \\ &= \sum_{\bar{v}_{\mathcal{V} \setminus V_i}} \prod_{\phi \in N(V_i)} F_\phi(\bar{v}) \\ &= \prod_{\phi \in N(V_i)} \sum_{\bar{v}_{\text{subtree}(\phi)}} F_\phi(\bar{v}) \\ &= \prod_{\phi \in N(V_i)} \mu_{\phi \to V_i}(v_i) \end{aligned}$$

  where F_ϕ is the product of all the factors in the subtree attached to factor ϕ.

- We write $\sum_{\bar{v}_{\mathcal{V} \setminus V_i}}$ to be the summation over all assignments of the variables in $\mathcal{V} \setminus V_i$.
- Recursive algorithm passes messages from leaves up to root, and then back down again

Factor-to-variable messages

$\mu_{\phi \to V}(v)$ expresses the ϕ subtree's preference over the value v for variable V.
Let $N(\phi)$ be the set of variables connected to factor ϕ.

$$\mu_{\phi \to V}(v) = \sum_{\bar{a} \in \Omega_{N(\phi) \setminus V}} \phi(v, \bar{a}) \prod_{W \in N(\phi) \setminus V} \mu_{W \to \phi}(\bar{a}_W)$$

Base case if ϕ is a leaf:

$$\mu_{\phi \to V}(v) = \phi(v)$$

Think of $\mu_{\phi \to V}$ as representing $P(V)$ if all subtrees except ϕ were cut off.
Note: $\prod$ is looping over children variables, and $\sum$ is looping over all assignments of children variables.

Variable-to-factor messages

$\mu_{V \to \phi}(v)$ expresses the V subtree's preference over the value v for variable V.

$$\mu_{V \to \phi}(v) = \prod_{\psi \in N(V) \setminus \phi} \mu_{\psi \to V}(v)$$

Base case if V is a leaf:

$$\mu_{V \to \phi}(v) = 1$$

Think of $\mu_{V \to \phi}$ as representing $P(V)$ if factor ϕ were cut off.

Sum-Product: marginal at root

Think in terms of operations on factors, where

- $\sum_{V_1,\ldots V_k} \phi$ means the factor that remains when we marginalize variables $V_1, \ldots, V_k$ out of ϕ and where
- $\phi_1 \times \ldots \times \phi_k$ means the factor we get by performing "table multiplication" on $\phi_1, \ldots, \phi_k$.

1. Select V_i as root
2. Recursively compute $P(V_i) \propto \prod_{\phi \in N(V_i)} \mu_{\phi \to V_i}$
3. In terms of

$$\mu_{\phi \to V} = \sum_{W \in N(\phi) \setminus V} \phi \prod_{W \in N(\phi) \setminus V} \mu_{W \to \phi}$$

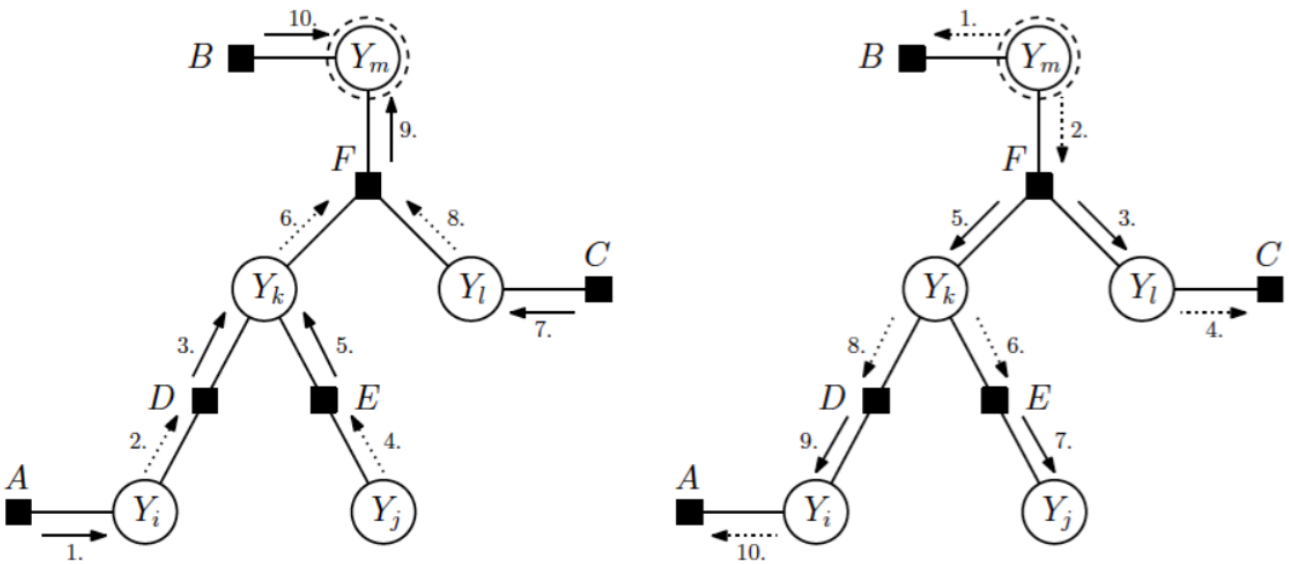
and

$$\mu_{V \to \phi} = \prod_{\psi \in N(V) \setminus \phi} \mu_{\psi \to V}$$

Sum-Product for all marginals

First, recursively traverse tree to compute "upward" messages
Then, pass messages back down to leaves, computing all marginals

$$P(V_j) \propto \prod_{\phi \in N(V_j)} \mu_{\phi \to V_j}$$



Recall that $\propto$ means "proportional to," and we generally need to normalize to get a distribution.

Handling evidence

To compute $P(V \mid E = e)$, add a new potential for every variable $V_i \in E$ that assigns 1 to $V_i = e_i$ and 0 to all other values for V_i. Then run sum-product.

More than marginal!

Easy to compute $P(V_i, V_j)$ if they are connected in the graph via one factor ϕ:

$$P(V_i, V_j) \propto \phi \prod_{\phi_i \in N(V_i) \setminus \phi} \mu_{\phi_i \to V_i} \prod_{\phi_j \in N(V_j) \setminus \phi} \mu_{\phi_j \to V_j} \prod_{V_k \in N(\phi) \setminus \{V_i, V_j\}} \mu_{V_k \to \phi}$$

Multiply everything coming into V_i, V_j, and ϕ from elsewhere, and normalize

If they aren't neighbors, then for each value $V_i = v_i$, compute

$$P(V_i = v_i, V_j = v_j) = P(V_i = v_i \mid V_j = v_j) P(V_j = v_j)$$

using tools we have already established.

Finding most probable assignment in a factor graph

We can an algorithm very similar to sum product, called max product. Just as $ab + ac = a \cdot (b + c)$,
$\max(ab, ac) = a \cdot \max(b, c)$ for non-negative a.
Do forward pass with messages as for sum-product, but

$$\mu_{\phi \to v}(v) = \max_{\bar{a} \in \Omega_{N(\phi) \setminus v}} \phi(v, \bar{a}) \prod_{w \in N(\phi) \setminus v} \mu_{w \to \phi}(\bar{a}_w)$$

Keep track of the values of w that yielded the max for each v:

$$M_v(v) = \operatorname*{argmax}_{\bar{a} \in \Omega_{N(\phi) \setminus v}} \phi(v, \bar{a}_w) \prod_{w \in N(\phi) \setminus v} \mu_{w \to \phi}(\bar{a}_w)$$

Decoding to find most probable assignment

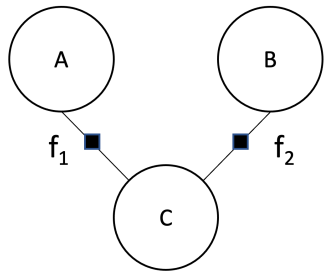
Work backward from root V:

$$v^* = \operatorname*{argmax}_v \mathsf{P}(v)$$

Best value for each child W_i of V:

$$w_1^*, \ldots, w_k^* = \mathsf{M}_V(v)$$

Sum-Product Practice



$$P(C) = \sum_A \sum_B P(A, B, C)$$
$$= 1/Z \prod_i \mu_{f_i \to C}$$
$$\mu_{f_1 \to C}(C) = \sum_A \phi_{f_1}(A, C)$$
$$\mu_{f_2 \to C}(C) = \sum_B \phi_{f_2}(B, C)$$

- Suppose

$$\phi_{f_1}(C, A) = \begin{bmatrix} A & C & \phi_{f_1}(A, C) \\ T & T & 0.05 \\ T & F & 0.45 \\ F & T & 0.45 \\ F & F & 0.05 \end{bmatrix}$$

Then

$$\mu_{f_1 \to C} = \begin{bmatrix} C & \mu \\ T & .5 \\ F & .5 \end{bmatrix}$$

- Suppose

$$\phi_{f_2}(B, C) = \begin{bmatrix} B & C & \phi_{f_2}(B, C) \\ T & T & 0.0 \\ T & F & 0.5 \\ F & T & 0.0 \\ F & F & 0.5 \end{bmatrix}$$

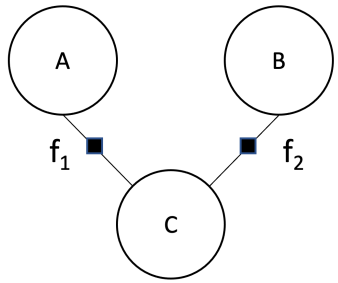
Then

$$\mu_{f_2 \to C} = \begin{bmatrix} C & \mu \\ T & 0 \\ F & 1 \end{bmatrix}$$

- Which means

$$P(C) = (1/Z)(\mu_{f_1 \to C} \times \mu_{f_2 \to C})$$
$$= 1/Z \begin{bmatrix} C & \mu \\ T & .5 \\ F & .5 \end{bmatrix} \times \begin{bmatrix} C & \mu \\ T & 0 \\ F & 1 \end{bmatrix}$$
$$= \begin{bmatrix} C & P(C) \\ T & 0 \\ F & 1 \end{bmatrix} \text{ where } Z = 0.5$$

Sum-Product Practice



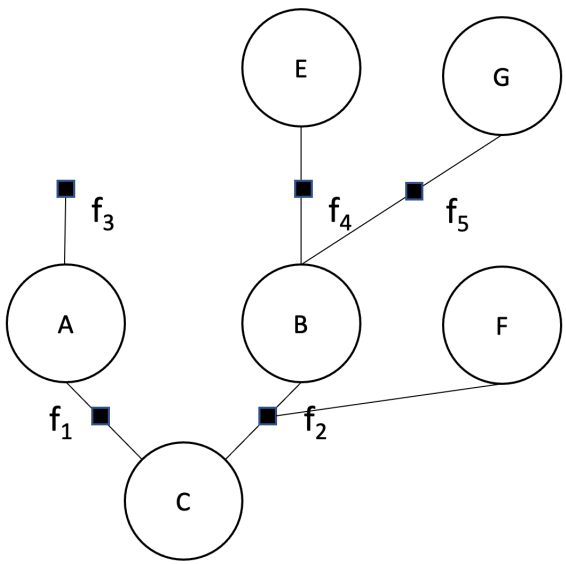
$$P(C) = 1/Z \prod_{i=1}^{2} \mu_{f_i \to C}$$

$$\mu_{f_1 \to C} = \sum_{A} \phi_{f_1} \mu_{A \to f_1}$$

$$\mu_{A \to f_1} = \mathbf{1}$$

$$\mu_{f_2 \to C} = \sum_{A} \phi_{f_2} \mu_{B \to f_2}$$

$$\mu_{B \to f_2} = \mathbf{1}$$



$$\mu_{f_3 \to A} = f_3$$

$$\mu_{A \to f_1} = \mu_{f_3 \to A}$$

$$\mu_{f_1 \to C} = \sum_{A} f_1 \cdot \mu_{A \to f_1}$$

$$\mu_{E \to f_4} = \mathbf{1}$$

$$\mu_{f_4 \to B} = \sum_{E} f_4 \cdot \mu_{E \to f_4}$$

$$\mu_{G \to f_5} = \mathbf{1}$$

$$\mu_{f_5 \to B} = \sum_{G} f_5 \cdot \mu_{G \to f_5}$$

$$\mu_{B \to f_2} = \mu_{f_4 \to B} \cdot \mu_{f_5 \to B}$$

$$\mu_{F \to f_2} = \mathbf{1}$$

$$\mu_{f_2 \to C} = \sum_{B,F} f_2 \cdot \mu_{B \to f_2} \cdot \mu_{F \to f_2}$$

$$P(C) \propto \mu_{f_1 \to C} \cdot \mu_{f_2 \to C}$$

Note that, to do the backward pass, you **do not** pass $P(C)$ back out. So $\mu_{C \to f_1} = \mu_{f_2 \to C}$. Similarly $\mu_{C \to f_2} = \mu_{f_1 \to C}$. And then $\mu_{f_2 \to F} = \sum_{C,B} f_2 \cdot \mu_{C \to f_2} \cdot \mu_{B \to f_2}$.