L21 – Discrete partially observable domains

AIMA 4.4, 17.4; KAlg 20.1-3; 21.1-2, 22.4-5

What you should know

- How to plan in belief space with set-based uncertainty
- How to model planning problems with a POMDP
- The Belief Space MDP
- Expectimax for partially observable problems
- Most-likely state and most-likely observation deterministic approximations

Non-observability

What if the robot does not know what state it is in and can't gain any information? We'll call this UOND (un-observable non-deterministic) planning.

- Could be uncertainty about the initial state
- Could be additional non-determinism in transitions that increase uncertainty
- If no possibility for observations, then we can try to find a conformant plan, which is guaranteed to succeed for all possible initial states and non-deterministic outcomes
- Do this through state-space search where now our states are belief states, which are sets of original states.

Reducing UOND planning to belief-space search

Given a non-observable-path problem $(\mathcal{S}, \mathcal{A}, \tilde{T}, G, C, S_o)$ where $\tilde{T}$ is non-deterministic as in FOND and $S_0 \subset \mathcal{S}$ is the set of possible starting states, we can generate a standard min-cost-path problem $(\mathcal{B}, \mathcal{A}', T', G', C, b_0)$ so that solution to the min-cost-path problem is a solution to the original non-observable problem:

- $\mathcal{B} = \text{powerset}(\mathcal{S})$ // Set of subsets of $\mathcal{S}$
- $\mathcal{A}' = \mathcal{A}$
- $b_0 = S_0$ // Single belief state is a set of env states
- $T'(b, a) = \bigcup_{s \in b} T(s, a)$
- $G' = \{b \mid b \subseteq G\} = \text{powerset}(G)$
- $C'(b, a, b')$ can really only depend on a

Important to note that even though transition on states $\tilde{T}$ is non-deterministic, the belief-space transition function T' <u>is deterministic</u>.

Adding observations

If we can make some observations, then we get the POND (partially observable non-deterministic) planning setting.

- Finite observation set $\mathcal{O}$
- perception function $O : \mathcal{S} \to \mathcal{O}$

It tells us what we will see in each state. For example, a vacuum robot with a local dirt detector might have $\mathcal{O} = \{\textit{clean}, \textit{dirty}\}$ and perception function telling it about its current location.

Includes completely observable and completely unobservable cases. How?

Use it to do a belief update:

- Given current belief (set of possible states) S and an actual observation o, what should we believe?
- Rule out states that could not have generated o:
- $\textsc{Update}(b, o) = \{s \in b \mid O(s) = o\}$

Searching in PO environments

And/Or search in belief space! Have to do **and** branches on the possible observations. Given non-deterministic, partially observable problem $(\mathcal{S}, \mathcal{A}, \mathcal{O}, \mathcal{S}_0, \tilde{T}, O, G, C)$ we can generate a non-deterministic observable problem (solvable by AO search):

- $\mathcal{B} = \text{powerset}(\mathcal{S})$ // Set of subsets of $\mathcal{S}$
- $\mathcal{A}' = \mathcal{A}$
- $b_0 = \mathcal{S}_0$
- $\tilde{T}'(b, a) = \{\textsc{Update}(\tilde{T}(b, a), o) \mid o \in \textsc{possible-percepts}(T(b, a))\}$
- $G' = \{b \mid b \subseteq G\} = \text{powerset}(G)$
- $C(b, a, b')$ can really only depend on a

What observations could we possibly get in belief state b?

$$\textsc{possible-percepts}(b) = \{O(s) \mid s \in b\}$$

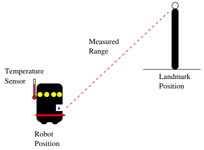
Agent for partially observable environment

- Initial belief $b = b_0$
- Make AO plan
- Take action a at root note
- Receive observation o from environment
- Update $b = \textsc{Update}(\tilde{T}(b, a), o)$
- Follow o branch in plan to get next action
- ...

Idea for later: In <u>receding-horizon</u> control, you can get away with making an approximate plan, and then replanning after every belief update.

Estimation and Planning

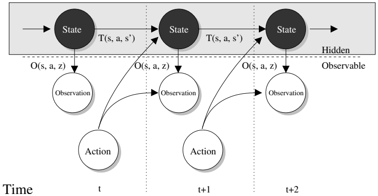
- Recall our motivating problem of a robot estimating its position.



- Things we know how to do:
  - Compute a distribution over the robot's position at time t, $P(x_t|z_t, a_t)$ based on a history of actions and observations, using message passing
  - Compute the robot's most likely trajectory from time 0 to t, $\hat{x}_{0:t} = \text{argmax}_{x_{0:t}} P(x_{0:t}|z_{0:t}, a_{0:t})$ based on a history of actions and observations, using Viterbi / max-sum
- What if the estimate of the robot's position is really noisy, so that $\hat{x}_t$ isn't particularly high probability?
- What if we had an action, such as "look at a map", that doesn't affect $\hat{x}_t$ but has a lot of impact on $P(x_t|z_{0:t}, a_{0:t})$?

Sequential decision-making with state uncertainty

- We are uncertain about the initial state
- Our goal remains to choose actions that maximize $E_{s_{0:T}}\left[\sum_t \gamma^t r(s_t)\right]$
- Since we don't know the state, future states and observations are not independent of past observations.



- Choice of action seems to depend on whole history of actions and observations
- We have to reason about the future observation sequences
- When the true state of the world cannot be observed directly (or is "inaccessible"), the world is "Partially Observable."

Partially Observed MDPs

MDP with added observation process

- MDP has
  - a set of states $\mathcal{S}$
  - a set of actions $\mathcal{A}$
  - transition model $T(s, a, s') = P(S_{t+1} = s' \mid S_t = s, A_t = a)$
  - reward function $R : \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}$
  - discount factor γ (and possibly horizon T)
- POMDP adds
  - a set of observations $\mathcal{O}$
  - observation model
    $O(s', a, o) = P(O_{t+1} = o \mid A_t = a, S_{t+1} = s')$
- optionally
  - an initial distribution over states, $b_0 = P(s_0)$

Belief states

The key to (somewhat) efficient action selection in POMDPs is the idea of a belief state.

- A belief state (or belief) b is a probability distribution over states
- Initial belief $b_0 = P(s_0)$ represents the agent's knowledge about the state when it starts.
- We define $b_t = P(s_t \mid a_0, o_0, \ldots, a_{t-1}, o_{t-1})$
- Computing b_t is a filtering problem.
- **Theorem**: b_t separates future states and observations from past actions and observations.
- So, if you know b_t you don't need to remember (or consider during decision-making) previous actions and observations.
- A policy for a POMDP can be described as a mapping from belief states to actions.

Recall Bayes filter belief update

Given previous belief b, action a and observation o, what is the new belief $b' = \mathsf{bf}(b, a, o)$?

$$\begin{aligned}
\mathsf{bf}(b, a, o)(s') &= P(S_{t+1} = s' \mid A_t = a, O_{t+1} = o, B_t = b) \\
&= \frac{1}{\eta} P(O_{t+1} = o \mid S_{t+1} = s', A_t = a, B_t = b) \cdot \\
&\quad\; P(S_{t+1} = s' \mid B_t = b, A_t = a) \\
&= \frac{1}{\eta} O(s', a, o) \sum_s P(S_{t+1} = s' \mid B_t = b, A_t = a, S_t = s) \cdot \\
&\quad\; P(S_t = s \mid B_t = b, A_t = a) \\
&= \frac{1}{\eta} O(s', a, o) \sum_s T(s, a, s') b(s)
\end{aligned}$$

where $\eta = P(o \mid a, b) = \sum_{\tilde{s}} O(\tilde{s}, a, o) \sum_s T(s, a, \tilde{s}) b(s)$

The Belief MDP

The process characterizing the way the beliefs change over time in a POMDP, is actually <u>an MDP in belief space</u>! Whoa!

- $\mathcal{S}' = \mathcal{B}$: In POMDP with n underlying world states, a belief $b \in \mathcal{B}$ is $(p_1, \dots, p_n)$ with $p_i = P(s_i)$, $0 \leqslant p_i$ and $\sum_i p_i = 1$
- $\mathcal{A}'$: Same as the POMDP
- $R' : \mathcal{B} \times \mathcal{A} \to \mathbb{R}$ where

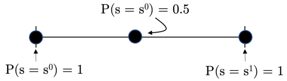
$$R'(b, a) = \sum_s R(s, a) b(s)$$

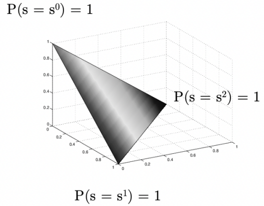
- $T' : \mathcal{B} \times \mathcal{A} \to \mathcal{B}$ where

$$T'(b, a, b') = \sum_{\{o \mid b' = \mathsf{bf}(b, a, o)\}} P(o \mid b, a)$$

  where $P(o \mid b, a) = \sum_s P(o \mid s, a) b(s) = \sum_s b(s) \sum_{s'} O(s', a, o) T(s, a, s')$

What exactly **is** Belief Space?





A simplex: the convex hull of the distributions where one of the states has probability one.

- For a two state MDP, this simplex is the line from 0 to 1.
    - At one end, the belief is that one of the states has probability 1.
    - At the other end, the belief is that the other state has probability 1.
    - All the points in between correspond to beliefs where the MDP might be in one of the two states, with different probabilities.
- For a three state MDP, the simplex is a 2D triangle embedded in 3D.
    - Each corner of the triangle corresponds to a belief where one of the states has probability 1.

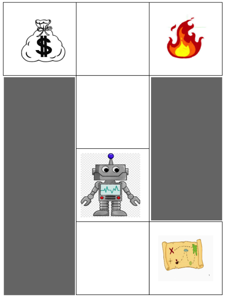
Formulation: Tiger domain

- Agent doesn't know which door the tiger lurks behind: prior is 50/50
- Agent can either listen to the tiger roar (-1), or open a door and escape ($+10$) or get eaten (-100).
- Listening is accurate with 85%.
- Tiger doesn't move until after we open a door, and escape or are eaten (world is stochastic).
  - What is the state space? What is $\mathrm{P}(s_0)$?
  - Suppose we listen once, and hear the tiger roar behind the left door. What is $\mathrm{P}(s_1)$? What action should we take next?

Problem forces the tradeoff between listening (gathering more information), or trusting the estimate

Formulation: Fortune or Ruin

- World is deterministic.
- Agent doesn't initially know which side of the T junction has $+100$ reward or -100 reward.
- Map at the bottom of the L describes which side has which reward with probability 1

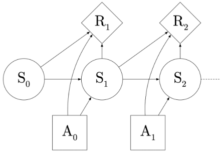


  - What is the state space?
  - What is $P(s_0)$?
  - Suppose the robot moves down one space. What is $P(s_t)$? What action should the robot take next?

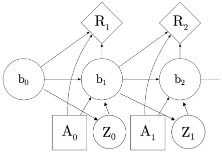
Problem forces the robot to plan to gather information.

Selecting actions in a belief-space MDP

Influence diagrams for an MDP and for a belief MDP:



Influence diagram for a Markov Decision Process (MDP)



Influence diagram for a belief state MDP

- The good news: can define and execute policies easily with respect to b_t
- The bad news: <u>computing</u> optimal policies exactly is <u>very</u> hard: will see that the worst-case computational cost is doubly-exponential in the horizon and $|O|$
- Two solution strategies:
  - On-line: expectimax, MCTS, approximations
  - Off-line: value-iteration, approximations

Online POMDP solution methods

Because the belief-space process is an MDP, we can use online MDP methods:

- Expectimax search:
  - Start search at current belief b_0
  - Branch on our choice of actions a
  - Branch on nature's choice of outcomes, which in a POMDP, is o
  - Result of doing a and observing o is $b' = \mathsf{bf}(b, a, o)$ with probability $\sum_s b(s) \sum_{s'} O(s', a, o) T(s, a, s')$
- Sparse sampling (expectimax with samples over o)
- Monte-Carlo Tree Search
- Determinization and other approximations: we outline a few choices

Very simple (bad) approximation: Most likely state

First, solve the underlying MDP to get π^{MDP}. Then

$$\pi^{\text{POMDP}}(b) = \pi^{\text{MDP}}(\operatorname*{argmax}_s b(s))$$

where π^{POMDP} is the policy of the POMDP, and π^{MDP} is the policy of the corresponding MDP that assumes the state is fully observable. What's wrong with this approach?

- Can't tell the difference between beliefs where the maximum-likelihood state is really certain (and just assuming it is the true state is reasonable) and where beliefs are really uncertain, and the true state could be anything.
- How would this approximation perform on Tiger?
- How would this approximation perform on Fortune-or-Ruin?
- **Key idea:** The uncertainty of the belief matters for the optimal policy. (We'll need a way to characterise the uncertainty.)
- Actually provably optimal for linear-quadratic-Gaussian systems.

Better approximation: QMDP

Assume the state will become fully observable on the next step. Solve the underlying MDP to get Q^{MDP}. Then

$$\pi(b) = \operatorname*{argmax}_a \sum_s b(s) Q^{\mathrm{MDP}}(s, a)$$

where Q^{MDP} is the Q-function of the corresponding MDP that assumes the state is fully observable.

- Takes into uncertainty into account for one step, but <u>only</u> one step, by hedging
- If the belief has high uncertainty, but there is some action that is not <u>too</u> bad across all the possible states, then that action will be preferred to some other action that is high risk/reward.
- What's wrong with this approximation? Tiger? Fortune?
- Hedging against uncertainty for one step may not be enough
- **Key idea:** The optimal policy may need to perform information gathering that incurs cost in order to resolve state uncertainty and ensure the agent can get large rewards (e.g., get to the goal with confidence).

Sometimes better approximation: Most-likely Observation

This time, we will turn our problem into a deterministic reward-maximization search problem in belief space.

- $\mathcal{S}' = \mathcal{B}$
- $s_0' = b_0$
- $R'(b, a, b') = R(b, a, b')$
- $T'(b, a) = \mathsf{bf}(b, a, o^*)$ where $o^* = \mathsf{argmax}_o\, P(o \mid b, a)$

This works very nicely for a POMDP version of a SSP problem, where instead of having an R', we can let the belief-space goal be something like believing with high probability that we are in a goal state.

$$G' = \left\{ b \,\middle|\, \sum_{s \in G} b(s) > .9 \right\}$$

MLO: strengths and weaknesses

- Requires replanning if ML observation is not obtained (or if resulting b is substantially different from the expected one)
- How would it work on Tiger or Fortune-or-ruin?
- How would this approach do on a linear-Gaussian MDP with continuous states?
    - Surprising result: the mean of this belief state is locally controllable, and the variance is deterministic.
    - Can plan directly in the configuration space, predict what the full beliefs will be, and compute the belief state costs.
    - Leads to very efficient planning performance.

Offline POMDP solution

Goal is to take a POMDP definition $\langle \mathcal{S}, \mathcal{A}, T, R, \mathcal{O}, O, \gamma \rangle$ and produce a policy that specifies what an agent should do, given any history of actions and observations.
Three ways of defining a policy:

- A policy tree of depth H is a conditional plan, which specifies an action to take for all observation/action histories.
- Mapping $\pi : \mathcal{B} \to \mathcal{A}$ combined with a module that does a Bayes filter to compute b_t from b_{t-1}, a_{t-1} and o_t.
- A set of α vectors, which is almost like a set of Q functions (except there might be more or fewer than $|\mathcal{A}|$), which implicitly defines $\pi : \mathcal{B} \to \mathcal{A}$.

Policy tree

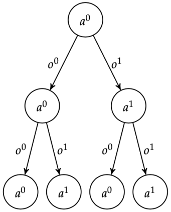


Figure 20.1. An example 3-step conditional plan.

Policy tree π (fig from KAlg):

- Nodes labeled with actions: action at root is $\pi.a$
- Arcs labeled with observations (have to include a child for every $o \in \mathcal{O}$: subtree of π for observation o is $\pi(o)$.

Value (expected discounted sum of rewards) of executing a policy tree π is a linear function of b.

Value of a policy tree: α vector

Let's start by computing the value of executing a policy tree π in a known starting state s: $V^\pi(s)$.

- Base case when π is single node ($H = 1$):

$$V^\pi(s) = R(s, \pi.a)$$

- Recursive case: depending on what state s' we transition to and what observation we get (which depends on the state), we will execute one of our subtrees ($\pi(o)$) in s':

$$V^\pi(s) = R(s, \pi.a) + \gamma \sum_{s'} T(s, \pi.a, s') \sum_{o} O(s', \pi.a, o) V^{\pi(o)}(s')$$

Let

$$\alpha^\pi = \left[V^\pi(s^1), \dots, V^\pi(s^{|\mathcal{S}|})\right]$$

Then value at a belief is

$$V^\pi(b) = \sum_s b(s) V^\pi(s) = b \cdot \alpha^\pi$$

Proof that $V = b \cdot \alpha$ (okay to ignore)

Base case when π is a single node:

$$V^{\pi}(b) = R(b, \pi.a) = \sum_{s} R(s, \pi.a) b(s) = \sum_{s} V^{\pi}(s) b(s)$$

Now assume for each child $V^{\pi(o)}(b) = b \cdot \alpha^{\pi(o)}$.

$$\begin{aligned}
V^{\pi}(b) &= R(b, \pi.a) + \gamma \sum_{o} P(o \mid b, \pi.a) V^{\pi(o)}(\mathsf{bf}(b, \pi.a, o)) \\
&= R(b, \pi.a) + \gamma \sum_{o} P(o \mid b, \pi.a) \mathsf{bf}(b, \pi.a, o) \cdot \alpha^{\pi(o)} \\
&= R(b, \pi.a) + \gamma \sum_{o} P(o \mid b, \pi.a) \sum_{s'} \mathsf{bf}(b, \pi.a, o)(s') \alpha^{\pi(o)}[s'] \\
&= R(b, \pi.a) + \gamma \sum_{o} P(o \mid b, \pi.a) \sum_{s'} \frac{\sum_{s} O(s', \pi.a, o) T(s, a, s')}{P(o \mid b, \pi.a)} \alpha^{\pi(o)}[s'] \\
&= \sum_{s} b(s) \left(R(s, \pi.a) + \gamma \sum_{o} \sum_{s'} O(s', \pi.a, o) T(s, a, s') \alpha^{\pi(o)}[s'] \right)
\end{aligned}$$

Stupidest possible finite-horizon optimal algorithm

- Enumerate all possible horizon H policy trees
  - How many policy trees are there?
  - Each horizon H policy tree has about $|\mathcal{O}|^H$ nodes
  - Each node can have $|\mathcal{A}|$ values
  - Total number of trees is about $|\mathcal{A}|^{|\mathcal{O}|^H}$
- Compute α^π for each one; call this set Γ_H
- Given an initial belief-state b, execute the policy tree with the highest expected value:

$$\operatorname*{argmax}_{\pi} b \cdot \alpha^\pi$$

The Tiger example – Formally

- Agent doesn't know which door the tiger lurks behind: prior is 50/50
- Agent can either listens to the tiger roar (-1), or open a door and escape ($+10$) or get eaten (-100).
- Listening is accurate with 85%.
- Tiger doesn't move until after we open a door, and escape or are eaten (world is stochastic).

- $\mathcal{S} = \{TL, TR\}$ tiger-left and tiger-right
- $\mathcal{A} = \{L, OL, OR\}$ listen, open-left, open-right
- Transition probabilities
  - $$P(s_i|a = L, s_j) = \begin{cases} 1 & \forall s_i == s_j \\ 0 & \text{otherwise} \end{cases}$$
  - open-left, open-right : transition to either tiger-left or tiger-right with probability 0.5
- $\mathcal{O} = \{HL, HR\}$ hear-left, hear-right
- Observation probabilities
  - After the listen action:

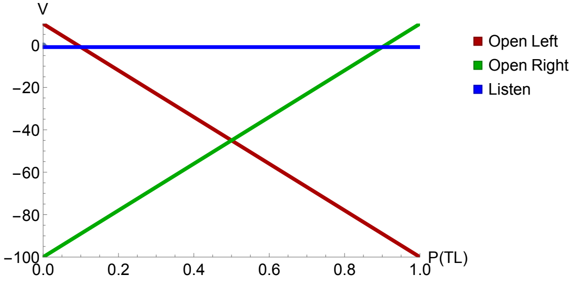
$$P(HR|a = L, S' = TR) = 0.85$$
$$P(HR|a = L, S' = TL) = 0.15$$
$$P(HL|a = L, S' = TR) = 0.15$$
$$P(HL|a = L, S' = TL) = 0.85$$

- Reward
  - $R(s_i, L) = -1$
  - $R(TL, OR) = 10$
  - $R(TR, OR) = -100$
  - $R(TL, OL) = -100$
  - $R(TR, OL) = 10$
- $\gamma = 0.75$.
- $b_0 : P(s_0 = TL) = 0.5$.

Tiger problem, simplified and solved!

Horizon 1: Policy trees are just a single node labeled by an action; $\alpha^a = [R(TL, a), R(TR, a)]$

- listen : $\alpha^L = [-1, -1]$
- open-left : $\alpha^{OL} = [-100, 10]$
- open-right : $\alpha^{OR} = [10, -100]$



Alpha vectors define policy and value function

Policy: In belief b, select the action at the root of policy tree π for which $\alpha^{\pi} \cdot b$ is maximal.

- When $P(TL) < .1$ then open left
- When $P(TL) > .9$ then open right
- Otherwise, listen

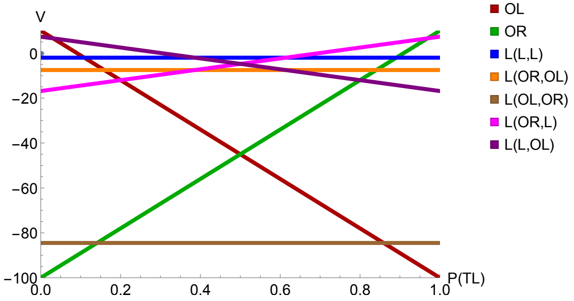
Value function: $V^*(b) = \max_{\alpha} b \cdot \alpha$

- Piecewise linear
- Convex

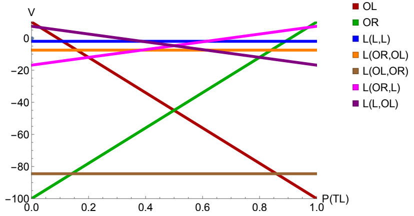
Horizon 2 tiger problem

For now, assume $\gamma = 1$ and after OL or OR there is no more reward. Let's consider three more policy trees, written A(B,C) where A is the root action, and if o = HL then do B otherwise do C.

- L(L, L) : $\alpha^{L(L,L)} = [-2, -2]$
- L(OL, OR) : $\alpha^{L(OL,OR)} = [-84.5, -84.5]$
- L(OR, OL) : $\alpha^{L(OR,OL)} = [-7.5, -7.5]$



Horizon 2 tiger problem: counterintuitive!



- Note that some policies trees are dominated: they are never the best choice. We can safely throw them away.
- Why is L(OR,OL) so bad? It seems like ought to be a good idea.
  - It is committed to always opening one of the doors.
  - Even if we start in a region of belief space where door-opening dominates, the first observation we get here might be wrong, and will cause us to the wrong action without any further consideration.
  - Better choices are L(OR, L) and L(L, OL)

Next time

- Exact value iteration
- Some more modern approximations, examples