L21: First-order Logic Intro

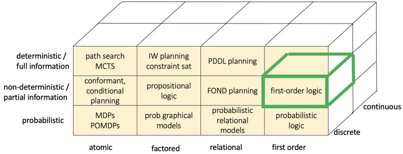
AIMA4e: Required: 8.1.2, 8.2
Suggested: 8.1, 8.3

What you should know after this lecture

- Definition of first-order logic: syntax and semantics
- What is FOL good for?
- How to formulate a FOL problem

Reasoning about object-based, open-world, partially-specified world states

Factored states
Boolean-valued factors
Objects as indices
Infinite domains



Note good section 8.1 in AIMA4e: talks about relationships among <u>different logics</u>, including probability theory and fuzzy logic.

What is first-order logic and what is it good for?

- Can model infinite possible worlds (environment states)
- Representation is lifted: we can abstract over objects in the world, describing them in terms of their properties and relations among them
- Contains propositional logic
- Inference procedures for determining the truth of some statement given the truth of others: semantics-preserving syntactic manipulation. Domain independent!

First-order logic syntax: symbols and terms

Three kinds of symbols. We will capitalize all of them. Both predicate and function symbols also have an arity (number of arguments they take).

- constant symbols: stand for objects
- predicate symbols: stand for relations on objects
  predicates with 0 arity are propositional symbols; include **True** and **False**
- function symbols: stand for functions

Terms stand for objects:

- constant symbols are terms
- variables are terms (we use lowercase letters)
- if ϕ is an arity-k function symbol and $\tau_1, \ldots, \tau_k$ are terms, then $\phi(\tau_1, \ldots, \tau_k)$ is a term

FOL syntax: sentences

- if π is an arity-k predicate symbol, and $\tau_1, \ldots, \tau_k$ are terms, then $\pi(\tau_1, \ldots, \tau_k)$ is a sentence
- if τ_1 and τ_2 are terms, then $\tau_1 = \tau_2$ is a sentence
- if α is a sentence and γ is a variable, then
  - $\forall \gamma . \alpha$ is a sentence
  - $\exists \gamma . \alpha$ is a sentence
- if α and β are sentences then
  $\neg \alpha$, $\alpha \vee \beta$, $\alpha \wedge \beta$, $\alpha \Rightarrow \beta$ and $\alpha \Leftrightarrow \beta$
  are sentences

FOL semantics: models

Model $m = (\mathcal{U}, \mathcal{I})$ a universe and an interpretation

- $\mathcal{U}$ is a (possibly infinite) set of objects
- If σ is a constant symbol, $\mathcal{I}(\sigma) \in \mathcal{U}$.
- If σ is an arity-k predicate symbol, $\mathcal{I}(\sigma) \subseteq \mathcal{U}^k$.
- If σ is an arity-k function symbol, $\mathcal{I}(\sigma) : \mathcal{U}^k \to \mathcal{U}$.

Arity-0 predicates are proposition symbols. How does that work?

Answer: $\mathcal{I}(\sigma)$ is a subset of $\mathcal{U}^0$. Formally, $\mathcal{U}^0 = \{(\ )\}$; that is, the set containing the empty tuple. This set has two possible subsets, so the interpretation of σ can either be $\{(\ )\}$ or $\{\ \}$.

FOL compositional semantics

Terms denote objects; we'll extend the use of $\mathcal{I}$:

- $\mathcal{I}(\phi(\tau_1, \ldots, \tau_k)) = \mathcal{I}(\phi)(\mathcal{I}(\tau_1), \ldots, \mathcal{I}(\tau_k))$

When is a sentence α is true in $m = (\mathcal{U}, \mathcal{I})$?

- Sentence $\pi(\tau_1, \ldots, \tau_k)$ is true in m iff $(\mathcal{I}(\tau_1), \ldots, \mathcal{I}(\tau_k)) \in \mathcal{I}(\pi)$
- Sentence $\tau_1 = \tau_k$ is true in m iff $\mathcal{I}(\tau_1) = \mathcal{I}(\tau_k)$
- Sentence $\forall\gamma.\alpha$ is true in m iff for all $o \in \mathcal{U}$, the sentence α holds in m extended by $\{\gamma/o\}$.
- Sentence $\exists\gamma.\alpha$ is true in m iff there exists some $o \in \mathcal{U}$, for which the sentence α holds in m extended by $\{\gamma/o\}$.
- Cases for the propositional connectives are as in PL

Practice

1. All cats are big.
2. There is a small thing or a cat.
3. All big things are cats.
4. Everything is a big cat.
5. All cats are small.
6. There is a big cat.
7. All small things are cats.
8. There is a big thing or a cat.

Example application: Airfare rules

Ontology: passenger, flight, city, airport, terminal, flight segment (list of flights, all in one day), itinerary (passenger + list of flight segments), list, number

Predicates: Age, Nationality, Wheelchair, Origin, Destination, Departure-Time, Arrival-Time, Latitude, Longitude, In-country, In-City, Itinerary, Nil

Function: Cons; so Cons(A, Cons(B, Nil)) is a list with 2 elts

Well-formed itineraries

A trip is well-formed iff all of the departures and arrivals match up (for now, the same airport, though one could relax it to be the same city or metro area) and all the layovers are legal.

$$\textit{Well-formed}(\mathbf{Nil})$$

$$\forall \mathsf{f}.\textit{Well-formed}(\textit{Cons}(\mathsf{f}, \mathbf{Nil}))$$

$$\forall \mathsf{f}_1, \mathsf{f}_2, \mathsf{r}.\ \textit{Contiguous}(\mathsf{f}_1, \mathsf{f}_2) \wedge \textit{Layover-legal}(\mathsf{f}_1, \mathsf{f}_2) \wedge$$
$$\textit{Well-formed}(\textit{Cons}(\mathsf{f}_2, \mathsf{r})) \Rightarrow \textit{well-formed}(\textit{Cons}(\mathsf{f}_1, \textit{Cons}(\mathsf{f}_2, \mathsf{r})))$$

Flight practice

1. Two flights are contiguous when the destination of the first is the same as the origin of the second.
2. A layover between two flights is legal if it's not too short and not too long.
3. You need at least 30 minutes to change planes.
4. A layover between f_1 and f_2 is not too long if it's less than 3 hours, or if there is no other flight leaving this aiport for the next immediate destination before f_2 departs.

Example application: Protocol verification

Elevators this time!

1. Between the time an elevator is called at a floor and the time it opens its doors at that floor, the elevator can arrive at that floor at most twice.
2. The cabin never moves with its door open.
3. Whenever the nth floor's call button is pressed, the cabin will eventually stop at the nth floor and open the door.

Using finite models for verification

One simple strategy for working with first-order theories is to test for bugs in smaller models.

- Given something like a cryptographic protocol or file system with unboundedly many objects.
- Make a finite-sized instance with constant symbols $\mathcal{C} = \{O_1, \ldots, O_k\}$
- Convert all quantifiers:
  - Change $\forall x.\alpha(x)$ to $\alpha(O_1) \wedge \ldots \wedge \alpha(O_k)$
  - Change $\exists x.\alpha(x)$ to $\alpha(O_1) \vee \ldots \vee \alpha(O_k)$
- Add an assertion that there is a bug (e.g., that two users can modify a file at the same time, or two trains can be on the same track segement)
- Use a SAT solver to see if the whole thing is satisfiable. If so, you found a bug!
- If no bug in an instance with a universe of size k you cannot say anything about what will happen with a larger universe.

More applications

- AWS proves whether you should have access to some particular asset.
- Business rules
- Hardware verification
- Crypto protocol verification
- Etc.

More kinds of logic

- Non-boolean valued: probability, fuzzy, trinary
- Modal:
  - Temporal: always, until, eventually, ...
  - Alethic: necessary, possible
  - Deontic: obligatory, permitted
  - Epistemic: $K(a, \phi)$ (agent a knows that ϕ)
- Special purpose (usually with efficient inference procedures)
  - Description logic (basically, taxonomies)
  - Reasoning about regular expressions

Entailment practice

Consider the following set of axioms:

- $\forall x. p(x) \leftrightarrow \exists y. r(x, y)$
- $\exists x, y. p(x) \wedge p(y) \wedge x \neq y$
- $\forall x, y. r(x, y) \rightarrow \neg r(y, x)$
- $\forall x, y, z. r(x, y) \wedge r(x, z) \rightarrow y = z$
- $\forall x, y, z. r(x, y) \wedge r(z, y) \rightarrow x = z$
- $\neg p(A)$

1. These axioms all hold in only two interpretations with the universe $U = \{A, B, C\}$. What are they?
2. Do these axioms entail sentence $\exists x. r(x, A)$?

Next time

- Syntactic proof